Ergodic theory of stochastic Petri networks

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ERGODIC THEORY OF STOCHASTIC PETRI NETWORKS

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Abstract

Stochastic Petri Nets are a general formalism for describing the dynamics of Discrete Event Systems. The present paper focuses on a subclass of Stochastic Petri Nets called Stochastic Event Graphs. Under the assumptions that the variables used for the timing of an Event Graph form stationary and ergodic sequences of random variables, we make use of an associated stochastic recursive equation in order to construct its stationary and ergodic regime. In particular, we determine the conditions under which the existence of this regime is guaranteed.

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Théorie ergodique des réseaux de Petri stochastiques

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Résumé

Les réseaux de Petri stochastiques offrent un formalisme général permettant de décrire la dynamique des systèmes à événements discrets. Cet article concerne une classe de réseaux de Petri stochastiques appelés graphes d'événements. Lorsque les suites de variables aléatoires de temporisation, sont stationnaires et ergodiques, on utilise une équation récursive stochastique associée au graphe d'événement pour construire son régime stationnaire et ergodique. En particulier, on détermine les conditions sous lesquelles l'existence d'un tel régime est garantie.

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1 Introduction

Timed Petri Networks can be viewed as a general formalism for describing the dynamics of Discrete Event Systems. It is beyond the scope of the present paper to review the domains of application of this formalism and the interested reader should for instance refer to the recent special issue of the *Proceedings of the IEEE* on Discrete Event Systems [9] for some entry points into the relevant literature.

It is nevertheless worthwhile mentioning that, similarly to other general formalism like Matthes Schemas (see [7] and [1]), or Generalized Semi Markov Processes ([13], [14] and [8]), this formalism is powerful enough for allowing one to describe virtually all existing models in Queueing Theory. Hence, this paper can first be seen as a direct continuation of Queueing Theory in that the subclass of Timed Petri Nets it focuses on, namely the class of Event Graphs, contains several classical models of Queueing Theory like single server queues, queues in tandem, closed cyclic networks, synchronized queueing networks as defined in [3] etc. In particular, the results obtained in the present paper allow one to solve a certain number of open problems of the Queueing Theory literature like for instance the existence of a stationary regime for closed cyclic networks under stationary service times assumptions. However, we will not review all the specific Queueing Theory problems for which the proposed approach leads to interesting results, for the primary aim of the present paper is to state structural results that hold within the more general formalism of Petri Nets.

Technically speaking, the main practical concerns of the paper are the construction of the stationary behavior of stochastic Event Graphs and the conditions under which such a stationary regime exists, when taking simple stationarity and ergodicity assumptions on the sequences used in the *timing* of the Event Graph. The two main probabilistic tools that are used, namely Ergodic Theory and Palm Probabilities stem from the analysis of stationary queues. Ergodic Theory has been used fruitfully in for more than twenty five years (see for instance [12], [5], [7] and [1]) in the context of Queueing Theory. As for Palm probabilities, see [7] and [1]. However, the Pointwise Ergodic Theorem, which is the very basis for analyzing classical queueing systems within this context, is not sufficient for the more general formalism of Petri Nets. It turns out that the adequate tool is a proper generalization of classical Ergodic Theory known as *Subadditive Ergodic Theory* ([10]).

The paper is structured as follows. Timed Petri Nets and the subclass of Timed Event Graphs are described together with some of their basic properties in §2. The basic equations that govern the evolution of such systems are established in §3 and the notion of *stable* Stochastic Timed Event Graph is introduced in §5. In §4, the Event Graph is decomposed into a set of strongly connected components, which are the vertices of some acyclic *Reduced Graph*. The minimal elements of this graph play a role similar to *sources* (i.e. elementary stationary point processes) in Queueing Theory. The construction of their stationary regime is the object of §6. §7 focuses on the ergodicity condition and the construction of the stationary regime of non-minimal elements of this Reduced Graph, which turn out to play a role similar to *servers* in Queueing Theory.

To the author's best knowledge, all the results presented in §6 and 7 are new, although the construction of §7 is partly based on a recent paper by the author ([2]).

Several interesting problems stem from our analysis and remain open for the time being, mainly concerning the so called minimal elements. For instance, these elements admit several different stationary regimes. §6 is in fact limited to the analysis of some particular stationary regime and nothing is known on the other stationary regimes. Similarly, the nature of the convergence of the transient behavior of a minimal element towards its steady state is not known. A particularly interesting question concerns the conditions under which the property of *convergence with coupling*, which holds for non minimal elements, also applies to minimal ones. Sufficient conditions for this property to hold are given in [4], mainly in the Markovian case. The construction of the present paper will be made on the stationary space of the minimal elements so as to avoid this coupling problem.

2 Stochastic Event Graphs

2.1 Petri Nets

A Petri Net is characterized by a triple (P, T, Γ), where P is the set of places, T the set of transitions and $\Gamma = (V, E)$ a directed graph with vertices $V = P \bigcup T$ and edges E. The edges of Γ are either of the form (p, t) or of the form (t, p) with $p \in P$ and $t \in T$. We shall denote by

- $\gamma^-(p)$ the set of transitions that precede place p in Γ: $\gamma^-(p) = \{t \in T \mid (t,p) \in E\}$;
- $\gamma^+(p)$ the set of transitions that follow place p in Γ: $\gamma^-(p) = \{t \in T \mid (p,t) \in E\}$;
- $\gamma^-(t)$ the set of places that precede transition t in Γ: $\gamma^-(t) = \{p \in P \mid (p,t) \in E\}$ and
- $\gamma^+(t)$ the set of places that follow transition t in Γ: $\gamma^+(t) = \{p \in P \mid (t,p) \in E\}$.

Tokens circulate in the Petri Net. This circulation takes place when transitions are fired. Two functions r and $s : P \times T \rightarrow I\!N$ are used to describe this mechanism:

- $r(p, t)$, the so called backward-incidence function, gives the number of tokens that are consumed by the firing of transition t in place $p \in \gamma^-(t)$.
- $s(p, t)$, the so called forward-incidence function, gives the number of tokens that are created by the firing of transition t in place $p \in \gamma^+(t)$.

Roughly speaking, the evolution of a Petri Net (P, T, Γ), of incidence functions r and s, is specified by the following rules: An initial condition is first given , the so called initial marking, which specifies the number of tokens initially present in each place. A transition t fires as soon as for each place p of $\gamma^-(t)$ there are at least $r(p, t)$ tokens present in the place. The firing of transition t consumes exactly $r(p, t)$ tokens in place p for all $p \in \gamma^-(t)$ and creates $s(p, t)$ tokens in place p for all $p \in \gamma^+(t)$.

2.2 Event Graphs

An Event Graph ([6]) is a Petri Net in which

- Each place is preceded and followed by exactly one transition: For, each $p \in P$, $|\{t \mid (t,p) \in E\}| = 1$ and $|\{t \mid (p,t) \in E\}| = 1$, where $|A|$ denotes the cardinality of the set A.
- The firing of a transition consumes exactly one token in each of the places that precede this transition and produces exactly one token in each of the places that follow it: For each $t \in T$, $r(p,t) = 1$ for all $p \in \gamma^-(t)$ and $s(p,t) = 1$ for all $p \in \gamma^+(t)$.

2.3 Timed Event Graphs

Generally speaking, the timing of an Event Graph is based on the following two rules:

- A token arriving at a place p cannot be consumed immediately by the transition that follows p. It stays there for a minimal amount of time, which will be called its service time in the sequel, and which may depend upon the place and the token. It is only when this service time is completed that the token can possibly be consumed by the transition that follows p.
- Transition t is enabled as soon as at least one token has completed its service in each of the places that precede t. Each transition starts firing as soon as it is enabled. This firing consumes one token from each of the places that precede t. The time interval between the epoch when transition t starts firing and the epoch when it completes firing (i.e. when the tokens produced by t are sent to the places that follow t) takes an amount of time, its firing time, that may depend upon t and the index of the firing.

Let $M(p)$ denote the number of tokens in the initial marking of place p. By definition, the n-th token, $n \geq 1$ to enter p will be one of these initial tokens if $n \leq M(p)$ and the $n - M(p)$-th to enter p after the beginning of the network evolution otherwise.

A Timed Event Graph will be said to be FIFO if

1. The token of place $p \in \gamma^-(t)$ consumed by the n-th ($n \geq 1$) firing of transition t is precisely the n-th token to enter place p.
2. The completion of the n-th firing of transition t produces the $M(p) + n$-th ($n \geq 1$) token arrival into place $p \in \gamma^+(t)$.

For instance, these conditions are satisfied if the service times of place p are such that tokens cannot overtake one another in the place (for instance when these service times are constant) and when the topology of the Event Graph is such that the $n+1$-st firing of a transition can only start after the completion of the n-th firing.

A transition t is *recycled* if there exists a place p with initial marking $M(p) = 1$ such that $\gamma^+(p) = \gamma^-(p) = t$. Observe that under the assumption that a transition is recycled, its $n+1$-st firing cannot start before the completion of the n-th one.

For example, a TEG with constant service times and recycled transitions is always FIFO, whatever the firing times sequences.

The following assumption will be made in the sequel:

H-1

The TEG under consideration is FIFO and has all its transitions recycled.

Another important remark is in order: For any Timed Event Graph (TEG), there exists an equivalent TEG with zero firing times. Indeed, consider a new TEG differing from the initial one only through its timing: In this new TEG, all transitions have zero firing times and for all transition t the service time of a token in a place of $\gamma^-(t)$ is taken to be equal to the sum of the corresponding service time in the initial TEG and the firing time of the transition that consumes it. Since each place is followed by exactly one transition, these new service times are defined unambiguously. Observe that for each place (resp. for each transition), the arrival times of tokens (resp. the ends of transition firings) take place exactly at the same times in both TEG's. When considering this type of state variables in a TEG, there is hence no loss of generality in assuming that firing is instantaneous.

In addition, one can always transform the initial TEG into another equivalent TEG where the initial marking has at most one token in each place. This is done by introducing additional places and transitions for each place with an initial marking having more than one token: for all place p such that $M(p) > 1$, let $q_2^p, \cdots, q_{M(p)}^p$ be $M(p) - 1$ new places with zero service times and $u_2^p, \cdots, u_{M(p)}^p$ be $M(p) - 1$ new transitions. On this set of vertices, the following additional edges are added to the initial ones: $\gamma^-(p) = u_2^p,\ \gamma_-(u_i^p) = q_i^p,\ i = 2, \cdots, M(p),\ \gamma^-(q_{M(p)}^p) = \gamma^-(p)$.

In the new TEG that is obtained after this transformation, consider the following initial marking: all places that had an empty marking initially receive an empty marking; each place p with $M(p) \geq 1$ initially receives one token; each new place receives one token in the new initial marking.

It is clear that each place in this new network contains at most one token in its initial marking, and that the sequences of firing times of the transitions that belong to both TEG's coincide, provided the places that belong to both TEG's have the same service times and the new places have zero service times. Observe that one can also add recycling places with zero service time and initial marking 1 to all the new transitions that were introduced without changing the time behavior of the system. Clearly, this new TEG depends both on the initial TEG and on the initial marking $M(p)$.

Finally, one can get rid of some unnecessary places as follows. For all pairs (s,t) of T, let

$$P(s,t) = \{p \in P \mid \gamma^+(p) = t \text{ and } \gamma^-(p) = s\} \tag{2.1}$$

One can always replace all the places of $P(s,t)$ by one single place with service times equal to the maximum of the service times of the places of $P(s,t)$. This place will be denoted by $p(s,t)$ in the sequel.

A FIFO and recycled TEG will be in its *canonical* form once the three preceding transformations have been performed. In its canonical version, the TEG satisfies the following properties.

1. *The transitions are all instantaneous and recycled.*
2. *The initial marking has at most one token in each place.*
3. *There are no pairs of transitions connected by several places.*

For sake of easy notation, we will also use the notation $\{P, T, \Gamma, M\}$ for this new TEG. This should not create any ambiguity since we will always work on this canonical form from now on.

2.4 Stochastic FIFO Event Graph

Let $\sigma_n^p \in IR^+,\ n \geq 1$ be the service time of the n-th token to enter place $p \in P,\ n \geq 1$. A Stochastic Event Graph is a Timed Event Graph where the service times $\{\sigma_n^p\}_{n=1}^{+\infty},\ p \in P$ are all Random Variables (RV's): more precisely , all results concerning properties of steady state will be based on the assumptions that the sequences $\{\sigma_n^p\}_{n=1}^{+\infty},\ p \in P$ are jointly stationary and ergodic sequences of non-negative and integrable RV's defined on a common probability space $(\Omega, F, \mathcal{P})$. The sequences $\{\sigma_n^p\}_{n=1}^{+\infty},\ p \in P$ can then be assumed to be the right half of certain bi-infinite sequences $\{\sigma_n^p\}_{-\infty}^{+\infty},\ p \in P$ on $(\Omega, F, \mathcal{P})$. Without loss of generality, we shall also assume that $(\Omega, F, \mathcal{P})$ is the canonical space of these $|P|$

sequences and denote by θ the left-shift operator on this space. As stated above, P will be assumed to be θ -invariant (stationary) and θ-ergodic. For reasons that will become apparent later on, it will also be assumed that the shift θ^N is ergodic for all integer $N \geq 1$. Within this framework, if one denotes by $\{\sigma^p\}$ the random variable $\{\sigma_0^p\}$, $p \in P$ we have then

$$\sigma_n^p = \sigma^p \circ \theta^n, \ p \in P, \ n \in \mathbb{Z}$$

For more detail on this formalism, see [BB 87].

3 Evolution Equations

This section concentrates on FIFO Stochastic Event Graph (FSEG) with recycled transitions in their canonical form.

3.1 Evolution of Firing Times

Let X_n^t, $t \in T$, $n \geq 1$ denote the time of the n-th firing of transition t, with the convention that for all $t \in T$, $X_n^t = \infty$ if transition t never fires for the n-th time. The initial condition of the FSEG is a non-negative vector Y^t $t \in T$, where Y^t represents the epoch when the initial token arrives into place p, $M(p) = 1$ and $t = \gamma^-(p)$, due to the firing of transition t that precedes X_1^t.

Observe that this definition of the initial condition implies the following constraints on Y:

$$0 \leq Y^t \leq X_1^t, \ t \in T \tag{3.1}$$

An initial condition that satisfies these constraints will be said to be *compatible*.

The variables of interest $\{X_n^t\}_{n=1}^{\infty}$, $t \in T$ satisfy the following evolution equations:

$$X_n^t = \ \max_{\{p \in \gamma^-(t)\}}\{X_{n-M(p)}^{\gamma^-(p)} + \sigma_{n-M(p)}^p\} \quad n = 1, 2, \cdots \tag{3.2}$$

where $X_0^t = Y^t$. Indeed, owing to Condition 1 of the FIFO assumption, the n-th firing, $n \geq 1$, of transition t takes place as soon as, for all $p \in \gamma^-(t)$, the n-th token to enter place p has completed its service there. In view of Condition 2 and of the compatibility assumption on the initial vector, this event takes place precisely at time $X_{n-M(p)}^{\gamma^-(p)} + \sigma_{n-M(p)}^p$, if $n > M(p)$ and at time $Y^{\gamma^-(p)} + \sigma_0^p$ otherwise.

The following observation will be of use later on. Assume that the initial condition is compatible and that each transition can fire at least once. Let $M'(p)$ be the marking of the network once each transition has fired exactly once. In view of the definition of Event graphs, $M'(p) = M(p)$, $\forall p \in P$ since the firing of $\gamma^-(p)$ produces exactly one token and the firing of $\gamma^+(p)$ consumes exactly one token. We can see this new marking as a new initial marking for the second run of transition firings. If the first initial condition is compatible, the token that is consumed by the first firing of $\gamma^+(p)$ is the one that was put into place p at time Y_0^p, so that the initial condition of the new initial marking is the vector X_1^t, $t \in T$. Observe that this new initial condition is also compatible since by definition $X_2^t \geq X_1^t, t \in T$.

3.2 Liveness of the Event Graph

The first natural question concerning the evolution of the FSEG is whether its transitions fire a finite or an infinite number of times, or equivalently whether the evolution equation 3.2 allows one to define the value of $X_n^t < \infty$ for all $t \in T$ and $n \geq 1$. Indeed, it is easy to construct simple examples of Event Graphs such that, for certain or all values of the initial marking, some transitions are never enabled, whatever the values of the service times.

This question is closely related to the so called *liveness* of the Petri Net ([6]). In our formalism, this problem boils down to the evaluation of the probability of the event

$$\bigcap_{n \geq 1} \{X_n^t < \infty\} \tag{3.3}$$

for each transition $t \in T$. In particular, if for all $n \geq 1$, $t \in T$, $X_n^t < \infty$ almost surely (a.s.), the FSEG will be said to be a.s. live. Since we are interested in the long term (and eventually the stationary) behavior of FSEG's, we will limit our attention to a.s. live networks. In Lemma 2, we provide a sufficient condition for this property to hold.

The following assumptions will be adopted in the sequel:

H-2

1. *The graph Γ is connected, in the sense that the underlying non-directed graph is connected.*
2. *For all $p \in P$, $\gamma^-(p) \neq \emptyset$.*

Observe that assumptions H-2.1 and H-2.2 introduce no loss of generality. A FSEG that does not satisfy H-2.1 can be decomposed into FSEG's of smaller dimension. A transition t that follows a place p without input transitions fires at most $M(p)$ times, so that a network that does not satisfy H-2.3 cannot be live.

Observe that the assumption that each transition is recycled entails the following additional property:

$$\forall t \in T, \ \gamma^-(t) \neq \emptyset \tag{3.4}$$

Before giving the main results on liveness, we introduce a new notion that will be needed in the sequel. Let $\mathcal{R}$ be the binary relation on $\mathcal{V} = \{(t,n), \ t \in T, \ n \in \mathbb{Z}\}$ defined by

$$(s,m)\mathcal{R}(t,n) \quad \text{iff} \quad \{p \in \gamma^-(t) \mid s = \gamma^-(p), \ M(p) = n - m\} \neq \emptyset. \tag{3.5}$$

Consider the directed graph $\{\mathcal{G} = \mathcal{V}, \mathcal{E}\}$ where

$$(s,m) \rightarrow (t,n) \in \mathcal{E} \quad \text{iff} \quad (s,m)\mathcal{R}(t,n). \tag{3.6}$$

In the sequel, $\pi(t,n)$, $t \in T$, $n \geq 1$ will denote the set of predecessors of node (t,n) of $\mathcal{G}$:

$$\pi(t,n) = \{(s,m) \in \mathcal{V} \mid (s,m)\mathcal{R}(t,n)\}. \tag{3.7}$$

Owing to assumption H on Γ,

$$\text{for all } (t,n),\ t \in T,\ n \geq 1, \quad \pi(t,n) \neq \emptyset. \tag{3.8}$$

Indeed, owing to Equation (3.4), for all $t \in T$, $\exists p \in \gamma^-(t)$ and from H-2.2, $\exists s = \gamma^-(p)$. Then, $(s, n - M(p))\mathcal{R}(t,n)$.

We are now in a position to state the liveness criterion, which is given in Lemmas 1 and 2 below.

Lemma 1 *Under the foregoing assumptions, $\mathcal{G}$ is an acyclic graph iff for each cycle in Γ, the initial marking $M(p)$, $p \in P$ has at least one place p with $M(p) > 0$.*

Proof

A cycle in $\mathcal{G}$ is equivalent to the existence of a sequence of say $q+1 \geq 2$ nodes of $\mathcal{V}$ that will be denoted by $(t_0,n_0),(t_1,n_1),\cdots,(t_q,n_q),(t_0,n_0)$, such that

$$(t_0,n_0)\mathcal{R}(t_1,n_1),\cdots,(t_q,n_q)\mathcal{R}(t_0,n_0). \tag{3.9}$$

From the very definition $(s,m)R(t,n)$ iff

(i) $n - M \leq m \leq n,\ n \geq 1$ and

(ii) $\{p \in \gamma^-(t) \mid s = \gamma^-(p),\ M(p) = n - m\} \neq \emptyset$.

Equation (3.9) together with (i) entail that $n_0 \leq n_1 \leq \cdots \leq n_q \leq n_0$, so that $n_0 = n_1 = \cdots = n_q$. Owing to (3.9), this and (ii) are in turn equivalent to the existence of a cycle in Γ where all the places are such that $M(p) = 0$, which completes the proof of the lemma. $\square$

For t and s in T, $n \geq 1,\ m \geq 0$ such that $(s,m)\mathcal{R}(t,n)$, consider the RV

$$\tau_m^{s,t} = \sigma_m^{p(s,t)}. \tag{3.10}$$

It is easily checked that Equations (3.2) can be rewritten as follows:

$$X_n^t = \max_{\{(s,m)\in\pi(t,n)\}} \{X_m^s + \tau_m^{s,t}\} \quad n = 1, 2, \cdots \tag{3.11}$$

Under the assumption that each cycle of Γ has a non-zero marking, we know from Lemma 1 that $\mathcal{G}$ is also acyclic, so that reexpressing recursively the terms of the type X_{n-1}^s in the RHS of Equation (3.11) by their value as given by (3.11) leads in at most $|T|$ steps to an expression where all terms in the RHS are of the form X_{n-1}^s, $s \in T$. More precisely, denote by $I^{s,t}$ the set

$$I^{s,t} = \{s = t_1, t_2, \cdots, t_m = t,\ 2 \leq m \leq |T| \text{s.t.} \quad (t_1, n)\mathcal{R}(t_2, n{+}1),\ (t_j, n{+}1)\mathcal{R}(t_{j+1}, n{+}1),\ j = 2, m- \tag{3.12}$$

Then, one can rewrite Equation (3.11) as

$$X_{n+1}^t = \max_{\{s\in T\}} \{X_n^s + \lambda_n^{s,t}\}, \quad n = 0, 1, 2, \cdots,\ t \in T \tag{3.13}$$

where $\lambda_n^{s,t} = -\infty$ if there is the set $I^{s,t}$ is empty and

$$\lambda_n^{s,t} = \max_{\{s=t_1,t_2,\cdots,t_m=t\in I^{s,t}\}} \{\tau_n^{t_1,t_2} + \tau_{n+1}^{t_2,t_3} + \cdots + \tau_{n+1}^{t_{m-1},t_m}\} \tag{3.14}$$

otherwise. Observe that $\lambda_n^{s,t}$ is non-negative and finite for all $n = 0, 1, \cdots$ such that $I^{s,t} \neq \emptyset$ and that for all $t \in T$ there is at least a $s \in T$ such that $I^{s,t} \neq \emptyset$.

In the sequel, the Equation (3.13) will be referred to as the *reduced form* of the evolution equation.

We are now in a position to prove the following liveness condition:

Lemma 2 *Under the foregoing assumptions, the FSEG is almost surely live if for each cycle in* Γ*, the initial marking* $M(p)$*,* $p \in P$ *has at least one place* p *with* $M(p) > 0$.

Proof

The condition is sufficient for liveness. The proof is by induction on n. We have $X_0^t = Y^t < \infty$ by assumption. Assume $X_n^t < \infty,\ t \in T$. We get then from Equation (3.13) that X_{n+1}^t is the maximum of at most $|T|$ finite RV's.$\square$

The following additional assumption will be adopted in the sequel:

H-3

The Event Graph under consideration is live.

The following additional property will be used later on.

Lemma 3 *Under the foregoing assumptions, any non-negative and finite initial condition is compatible.*

Proof

In view of the assumption that all transitions are recycled, it follows that for all $t \in T$, $\lambda_n^{t,t} \geq 0$ and it immediately follows from this and from Equation (3.13) that $X_1^t \geq Y_t,\ \forall t \in T$. $\square$.

In the sequel, we will also make use of the notion of *precedence graph* $\Gamma^* = \{V^*, E^*\}$ associated with Equations (3.13), which is defined as follows: the set of vertices V^* is the set of transitions T of the canonical FSEG. The set of edges E^* is the set of couples $\{s, t\}$ of T such that $\lambda^{s,t} \neq -\infty$ *a.s.*.

4 Decomposition of the Precedence Graph

Throughout this section, a FSEG will be assumed to be given in its canonical form (Equations (3.13). Consider now the decomposition of its precedence graph Γ^* into its maximal strongly connected subgraphs. Recall that a strongly connected graph is a directed graph in which the existence of a directed path from v_1 to v_2 implies the existence of another path from v_2 to v_1. A maximal strongly connected subgraph of a graph G is a strongly connected subgraph of G such that no other subgraph of G covering it is strongly connected. Let g be the number of the maximal strongly connected subgraphs in Γ^*, and $G_1 = (T_1, E_1)$, $\cdots$, $G_g = (T_g, E_g)$ be the set of all these subgraphs. It is easy to prove that the above set of subgraphs is uniquely defined and that

$$T_1 \bigcup \cdots \bigcup T_g = T, \tag{4.1}$$

$$E_1 \bigcup \cdots \bigcup E_g \subseteq E^*, \tag{4.2}$$

and for every i and j, $1 \leq i < j \leq g$,

$$T_i \bigcap T_j = \emptyset, \tag{4.3}$$

$$E_i \bigcap E_j = \emptyset. \tag{4.4}$$

If s and $t \in T$ belong to the same strongly connected subgraph of G, then certain firings of transition s and t are constrained by one another. On the other hand, if s and t belong to different strongly connected subgraphs, the constraint is oneway.

Define the Reduced Graph (RG) of G^*, which will be denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, to be the graph on $\mathcal{V} = \{1, 2, \cdots, g\}$ defined by the set of edges

$$\mathcal{E} = \{(h,k) | 1 \leq h, k \leq g, \ \exists s, t \in T, \ s \in \mathcal{G}_h, \ t \in \mathcal{G}_k \text{ and } (s,t) \in \ E^*\}$$

The RG describes the relations between the maximal strongly connected subgraphs of the FSEG. Obviously, $\mathcal{G}$ is acyclic. In the sequel, we shall distinguish between the minimal vertices of the RG, which will be denoted by $\{1, \cdots, g_0\}$, and the vertices that have predecessors in $\mathcal{G}$, which will be denoted by $\{g_0 + 1, \cdots, g\}$. In the sequel we will use the notation $\pi(h)$ (resp. $\pi^*(h)$) to represent the set of vertices of $\mathcal{G}$ that are direct (resp. direct or indirect) predecessors of h in $\mathcal{G}$. Without loss of generality, it will also be assumed that the numbering of the vertices is compatible with the graph in the sense that $(i, j) \in \mathcal{E}$ implies $i < j$.

5 Stable and Unstable Networks

Under the foregoing assumptions, a place of the FSEG will be said to be stable if the number of tokens in this place, also called its marking, converges weakly to a finite RV when time tends to ∞. The FSEG is said to be stable if all the network places are stable.

Consider the following partition of the set of places

$$P^0 = \{p \in P \mid \gamma^-(p) \text{ and } \gamma^+(p) \ \in T_h \text{ for some } 1 \leq h \leq g\} \tag{5.1}$$

and

$$P^1 = \{p \in P \mid \gamma^-(p) \ \in T_f, \ \gamma^+(p) \ \in T_h \text{ for some } 1 \leq f < h \leq g\} \tag{5.2}$$

P^0 is the set of places connecting two transitions belonging to a same strongly connected component, while P^1 is the set of places connecting transitions that do not belong to any cycle.

Lemma 4 *The marking of all places of P^0 is bounded, and the only places that can possibly have infinite marking when times goes to ∞ are those of P^1.*

Proof

Define a cycle of the Event Graph to be a sequence $p_1, t_2, p_2, \cdots, p_{n-1}, t_n$ where $\gamma^+(p_i) = t_{i+1}$, $1 \leq i < n$, $\gamma^-(p_i) = t_i$, $1 < i < n$, and $\gamma^-(p_1) = t_n$. It is easy to check that the total number of tokens in the places of any cycle of an Event Graph with zero firing times is invariant with time (use the fact that the firing of any transition of the cycle consumes exactly one token from the cycle and produces exactly one token into the cycle). The proof of the lemma follows from the fact that each place of P^0 necessarily belongs to a cycle. □

In the remaining sections we focus on of the conditions that must be satisfied by the statistics of the governing sequences $\{\sigma_n^p\}_{n=-\infty}^{\infty}$, $p \in P$ for ensuring the stability of the Event Graph, or equivalently, the asymptotic finiteness of the marking of the places in P^1.

6 Stationary Behavior of Minimal Elements of the Reduced Graph

It follows from Lemma 4 that all the places connecting transitions of a minimal strongly connected subgraph remain bounded with time. These places are then stable by nature. It is nevertheless necessary to analyze their stationary behavior for determining the stability condition of the other components.

Consider a maximal strongly connected subgraph G_h, $1 \leq h \leq g_0$. For sake of easy notation, we shall denote by $1, \cdots, K$ the transitions of T_h so that Equation (3.13) reads

$$\begin{aligned} X_0^k &= Y^k, & & 1 \leq k \leq K \\ X_{n+1}^k &= \max_{\{1 \leq j \leq K\}} \{X_n^j + \lambda_n^{j,k}\}, & & n = 0, 1, 2, \cdots, \ 1 \leq k \leq K \end{aligned} \tag{6.1}$$

In a first step, we get rid of the variables $\lambda_n^{j,k}$ that are equal to $-\infty$.

Consider the RV's

$$C_{n,n+m}^{j,k} = \max_{\{j=i_n, i_{n+1}, \cdots, i_{n+m}=k \in \{1,\cdots,K\}\}} \sum_{l=n}^{n+m-1} \lambda_l^{i_l, i_{l+1}}, \quad m = 1, 2, \cdots, \ n = 0, 1, \cdots \tag{6.2}$$

Let N be a fixed positive integer. Define

$$V_n^k = X_{nN}^k, \ n \geq 0 \tag{6.3}$$

One checks immediately that the variables V_n^k satisfy the recursion

$$\begin{aligned} V_0^k &= \quad Y^k, \quad 1 \leq k \leq K \\ V_{n+1}^k &= \max_{\{1 \leq j \leq K\}}\{V_n^j + \Lambda_n^{j,k}\}, \quad n = 0,1,2,\cdots, \ 1 \leq k \leq K \end{aligned} \tag{6.4}$$

where the variables $\Lambda_n^{j,k}$ are defined by

$$\Lambda_n^{j,k} = C_{nN,(n+1)N}^{j,k} \tag{6.5}$$

Owing to the strong connectedness assumption, there exists an integer N such that for all pair $1 \leq j,k \leq K$, one can find a sequence $j = i_0, i_2, \cdots, i_N = k$ in $\{1, \cdots, K\}$ for which $(i_{m-1}, i_m) \in E^*$, so that $\lambda_{m-1}^{i_{m-1},i_m} \neq -\infty \ \forall m = 1, \cdots, N$. For such N, one gets immediately that $\Lambda_n^{j,k}$ is in $I\!R^+$ (i.e. $\neq -\infty$). Furthermore, it is easily checked that $\Lambda_n^{j,k} = \Lambda^{j,k} \circ \theta^{nN}$, where $\Lambda^{j,k} = \Lambda_0^{j,k}$; and that $\Lambda^{j,k}$ is integrable (use that fact that $max(a,b) \leq a + b$).

We are now in a position to introduce the main state variables

$$W_n^{j,k} = V_{n+1}^k - V_n^j - \Lambda_n^{j,k} \quad , \ 1 \leq j,k \leq K, \ n = 0,1,2,\cdots \tag{6.6}$$

Denote by L_n the RV

$$L_n = \max_{1 \leq k,j \leq K} \Lambda_n^{j,k} \quad n = 0,1,2,\cdots \tag{6.7}$$

and by L the RV L_0.

Lemma 5 *For all* $1 \leq j,k \leq K, \ n = 1,2,\cdots$

$$0 \leq W_n^{j,k} \leq 2L_n + L_{n-1} \tag{6.8}$$

Proof

From Equation (6.4), $V_{n+1}^k \geq V_n^j + \Lambda_n^{j,k}$, which proves the leftmost inequality.

We focus now on the rightmost inequality. For $n \geq 1$, write

$$\begin{aligned}
\max_{1\leq j,k\leq K} W_n^{j,k} &\leq \max_{1\leq k\leq K} V_{n+1}^k - \min_{1\leq k\leq K} V_n^k \\
&= \max_{1\leq k\leq K} V_{n+1}^k - \min_{1\leq k\leq K} V_{n+1}^k \\
&+ \min_{1\leq k\leq K} V_{n+1}^k - \max_{1\leq k\leq K} V_n^k \\
&+ \max_{1\leq k\leq K} V_n^k - \min_{1\leq k\leq K} V_n^k
\end{aligned} \tag{6.9}$$

We have

$$\max_{1\leq k\leq K} V_{n+1}^k - \min_{1\leq k\leq K} V_{n+1}^k = \max_{1\leq k,j\leq K} \{V_n^j + \Lambda_n^{j,k}\} - \min_{1\leq k\leq K} \max_{1\leq j\leq K} \{V_n^j + \Lambda_n^{j,k}\}$$

$$\leq \max_{1\leq k\leq K} V_n^k + L_n - \min_{1\leq k\leq K} \max_{1\leq j\leq K} \{V_n^j + \Lambda_n^{j,k}\}$$

Owing to the fact that $\Lambda_n^{j,k} \geq 0$, we get

$$\min_{1\leq k\leq K} \max_{1\leq j\leq K} \{V_n^j + \Lambda_n^{j,k}\} \geq \max_{1\leq k\leq K} V_n^k$$

so that the bound

$$\max_{1\leq k\leq K} V_{n+1}^k - \min_{1\leq k\leq K} V_{n+1}^k \leq L_n, \quad n = 0, 1, 2, \cdots$$

is immediately obtained from the last relation.

Similarly

$$\min_{1\leq k\leq K} V_{n+1}^k - \max_{1\leq k\leq K} V_n^k \leq \min_{1\leq k\leq K} \max_{1\leq j\leq K} \{V_n^j + \Lambda_n^{j,k}\} - \max_{1\leq k\leq K} V_n^k$$

$$\leq \max_{1\leq k\leq K} \max_{1\leq j\leq K} \{V_n^j + \Lambda_n^{j,k}\} - \max_{1\leq k\leq K} V_n^k \leq L_n$$

Using these two bounds in Equation (6.9) allows one to complete the proof of Lemma 5. □

The basic evolution equation for the variables $W_n^{j,k}$ is given in the following lemma.

Lemma 6 *For all* $1 \leq j, k \leq K$,

$$W_0^{j,k} = \max_{1\leq i\leq K} \{Y^i + \Lambda_0^{i,k}\} - Y^j - \Lambda_0^{j,k} \tag{6.10}$$

and for $n = 1, 2, \cdots$

$$W_{n+1}^{i,j} = \max_{1\leq k\leq K} \min_{1\leq l\leq K} \{W_n^{l,k} + \Xi_n^{l,k,i,j}\} \tag{6.11}$$

where

$$\Xi_n^{l,k,i,j} = (\Lambda_n^{l,k} + \Lambda_{n+1}^{k,j}) - (\Lambda_n^{l,i} + \Lambda_{n+1}^{i,j}) \tag{6.12}$$

Proof

From the identity $\max(a,b) = a + (\max(a,b) - a)$, we get

$$V_{n+2}^j = V_{n+1}^i + \Lambda_{n+1}^{i,j} + \max_{1\leq k\leq K} \{V_{n+1}^k + \Lambda_{n+1}^{k,j} - \max_{1\leq l\leq K} \{V_n^l + \Lambda_n^{l,i}\} - \Lambda_{n+1}^{i,j}\}$$

so that

$$W_{n+1}^{i,j} = \max_{1\leq k\leq K} \min_{1\leq l\leq K} \{V_{n+1}^k + \Lambda_{n+1}^{k,j} - V_n^l - \Lambda_n^{l,i} - \Lambda_{n+1}^{i,j}\}$$

which completes the proof. □

As usual, we shall say that Equation (6.11) has a stationary solution if we can find non-negative and finite random variables $W^{i,j}$ such that

$$W^{i,j} \circ \Theta = \max_{1\leq k\leq K} \min_{1\leq l\leq K} \{W^{l,k} + \Xi^{l,k,i,j}\} \tag{6.13}$$

where $\Theta = \theta^N$ and

$$\Xi^{l,k,i,j} = (\Lambda^{l,k} + \Lambda^{k,j} \circ \Theta) - (\Lambda^{l,i} + \Lambda^{i,j} \circ \Theta) \tag{6.14}$$

In order to find such a solution, consider the sequence $M_n^{i,j}$, $n \geq 0, 1 \leq i, j \leq K$ defined by $M_0^{i,j} = 0$, and

$$M_{n+1}^{i,j} \circ \Theta = \max_{1 \leq k \leq K} \min_{1 \leq l \leq K} \{M_n^{l,k} + \Xi^{l,k,i,j}\}, \quad n = 0, 1, \cdots \tag{6.15}$$

We prove by induction that for all $1 \leq i, j \leq K$, the sequence $M_n^{i,j}$ is increasing in n. Observe first that $M_1^{i,j} \geq M_0^{i,j} = 0$. Assume now that $M_n^{i,j} \geq M_{n-1}^{i,j} = 0$ for all $1 \leq i, j \leq K$. Then

$$\begin{aligned} M_{n+1}^{i,j} \circ \Theta &= \max_{1 \leq k \leq K} \min_{1 \leq l \leq K} \{M_n^{l,k} + \Xi^{l,k,i,j}\} \\ &\geq \max_{1 \leq k \leq K} \min_{1 \leq l \leq K} \{M_{n-1}^{l,k} + \Xi^{l,k,i,j}\} \\ &= M_n^{i,j} \circ \Theta \end{aligned}$$

which completes the proof. Let denote by $M_\infty^{i,j}$ the limit

$$M_\infty^{i,j} = \lim_{n \to \infty} M_n^{i,j} \tag{6.16}$$

Lemma 7 *For all* $1 \leq i, j \leq K$,

$$0 \leq M_\infty^{i,j} < \infty \tag{6.17}$$

and is integrable. These variables satisfy the relation

$$M_\infty^{i,j} \circ \Theta = \max_{1 \leq k \leq K} \min_{1 \leq l \leq K} \{M_\infty^{l,k} + \Xi^{l,k,i,j}\} \tag{6.18}$$

Furthermore, M_∞ is the minimal solution of Equation (6.11) in the sense that any other solution W satisfies the relation

$$W^{i,j} \geq M_\infty^{i,j} \tag{6.19}$$

Proof

It follows from Lemma 5 that $N_n^{i,j} = W_n^{i,j} \circ \Theta^{-n}$, $n = 0, 1, \cdots$ satisfies the inequalities

$$0 \leq N_n^{i,j} \leq 2L + L \circ \Theta^{-1} \quad n = 1, 2, \cdots \tag{6.20}$$

In addition, N_n satisfies the relations

$$N_0^{j,k} = \max_{1 \leq i \leq K} \{Y^i + \Lambda_0^{i,k}\} - Y^k \geq 0 \tag{6.21}$$

and

$$N_{n+1}^{i,j} \circ \Theta = \max_{1 \leq k \leq K} \min_{1 \leq l \leq K} \{N_n^{l,k} + \Xi^{l,k,i,j}\} \quad n = 0, 1, \cdots \tag{6.22}$$

It follows from this and from an induction of the same type as above that

$$M_n^{i,j} \leq N_n^{i,j} \tag{6.23}$$

for all $n = 0, 1, \cdots$ and for all $1 \leq i, j \leq K$, so that the first property of the lemma follows immediately from Equations (6.20) and (6.23).

The integrability property follows immediately from the bound

$$M_\infty^{i,j} \leq 2L + L \circ \Theta^{-1} \tag{6.24}$$

and from the integrability of L.

Equation (6.18) follows from Equation (6.15) by letting n go to ∞.

Let W be another non-negative solution. We prove that

$$W^{i,j} \geq M_n^{i,j}$$

for all $1 \leq i, j \leq K$, $n = 0, 1, \cdots$. This is obvious for $n = 0$, since $M_0^{i,j} = 0$. Assume this is true up to rank n, then

$$W^{i,j} \circ \Theta \quad = \quad \max_{1 \leq k \leq K} \min_{1 \leq l \leq K} \{W^{l,k} + \Xi^{l,k,i,j}\}$$

$$
\begin{aligned}
&\geq \max_{1\leq k\leq K}\min_{1\leq l\leq K}\{M_n^{l,k}+\Xi^{l,k,i,j}\}\\
&= M_{n+1}^{i,j}\circ\Theta
\end{aligned}
$$

which concludes the proof. The first assertion of the lemma follows immediately from this inequality by letting n go to ∞.□

We focus now on the relation between the existence of a stationary solution for Equation (6.11) and the stationary behavior of the underlying FSEG.

Consider the variables

$$\delta_n^{i,j} = X_{n+1}^j - X_n^i, \quad n = 0, 1, 2, \cdots \tag{6.25}$$

and

$$\Delta_n^{i,j} = X_{(n+1)N}^j - X_{nN}^i, \quad n = 0, 1, \cdots \tag{6.26}$$

Owing to Lemma 7, for each minimal subnetwork of the Reduced Graph the sequence $\Delta_n^{i,j}$, $1 \leq i,j \leq K$ can be made stationary by an appropriate choice of the initial condition Y^k, $1 \leq k \leq K$.

Indeed, consider the system

$$M_\infty^{j,k} = \max_{1\leq i\leq K}\{Y^i + \Lambda_0^{i,j}\} - Y^j - \Lambda_0^{j,k} \tag{6.27}$$

where Y^k, $1 \leq k \leq K$ is the set of unknowns. One can get a solution of this system as follows: by subtracting, we get $Y^i - Y^k = M_\infty^{j,k} + \Lambda_0^{j,k} - M_\infty^{j,i} - \Lambda_0^{j,i}$. Let k_0 be such that $M_\infty^{j,k_0} + \Lambda_0^{j,k_0} \geq M_\infty^{j,k} + \Lambda_0^{j,k}, \forall 1 \leq k \leq K$. Take $Y^{k_0} = 0$ and $Y^i = M_\infty^{j,k_0} + \Lambda_0^{j,k_0} - M_\infty^{j,i} - \Lambda_0^{j,i}$. This (or any other nonnegative and finite) solution of this system is such that $W_n^{j,k} = M_\infty^{j,k}\circ\Theta^n$, $n \geq 0$ and puts the sequences $W_n^{j,k}$ and hence the sequences $\Delta_n^{j,k} = W_n^{j,k} + \Lambda_n^{j,k}$ in their steady state. We will continue the stationary sequence $\Delta_n^{j,k}$, $n \geq 1$ by the bi-infinite sequence $\Delta_n^{j,k}$, $n \in \mathbb{Z}$ by the formula

$$\Delta_n^{j,k} = \Delta_0^{j,k}\circ\Theta^n, \; n \in \mathbb{Z} \tag{6.28}$$

This sequence is also ergodic since the shift $\Theta = \theta^N$ is assumed to be ergodic.

As it is shown in the next theorem, it turns out that the variables $\delta_n^{i,j}$, $1 \leq i, j \leq K$ can similarly be made stationary and ergodic with respect to the initial shift θ.

Theorem 1 *With an appropriate choice of the initial condition Y^k, $1 \leq k \leq K$, the variables $\delta_n^{i,j} = X_{n+1}^j - X_n^i$, $i, j \in T_h$, $n \geq 0$, $1 \leq h \leq g_0$, are stationary and ergodic in the sense that they admit the following representation*

$$\delta_n^{i,j} = \delta^{i,j} \circ \theta^n, \quad n = 0, 1, \cdots \tag{6.29}$$

for some finite and integrable random variable $\delta^{i,j}$.

Proof.

We prove first the property for one single component G_h. It follows from the preceding construction that the variables

$$\Delta_n^{i,j} = \{M_\infty^{i,j} + \Lambda^{i,j}\} \circ \Theta^n \tag{6.30}$$

can be represented as

$$\Delta_n^{i,j} = \Delta^{i,j} \circ \Theta^n, \quad n \in \mathbb{Z}$$

for some finite random variable $\Delta^{i,j}$. Clearly, a similar property also holds for the variables

$$\Delta_n^{l,j} - \Delta_n^{l,i} = X_{nN}^j - X_{nN}^i \tag{6.31}$$

Observe that it is easy to reconstruct the variables $X_{nN+m+1}^k - X_{nN+m}^l$, $1 \leq k, l \leq K$, $0 \leq m \leq N$, from the knowledge of $X_{nN}^j - X_{nN}^i$, $1 \leq i, j \leq K$. For $n = 0$, we have indeed

$$X_1^k - X_0^i = \max_{\{1 \leq j \leq K\}} \{X_0^j - X_0^i + \lambda_1^{j,k}\}, \quad 1 \leq i, k \leq K \tag{6.32}$$

and

$$X_{n+1}^k - X_n^i = \max_{\{j \in T_h |\ (j,k) \in E^\bullet\}} \min_{\{l \in T_h |\ (l,i) \in E^\bullet\}} \{X_n^j - X_{n-1}^i + \lambda_n^{j,k} - \lambda_{n-1}^{l,i}\} \qquad (6.33)$$

for all $n = 1, \cdots, N-1,\ 1 \le i, k \le K$.

The sequence $\{X_{nN+m+1}^k - X_{nN+m}^l,\ 1 \le k, l \le K,\ 0 \le m < N\}_{n=-\infty}^{\infty}$ is hence stationary and ergodic with respect to Θ. In other words, $\{X_n^k - X_{n-1}^l,\ 1 \le k, l \le K\}_{n=-\infty}^{\infty}$ forms, on the probability space $(\Omega, F, \mathcal{P})$, a stochastic sequence with an embedded point process in its synchronous version, as defined by Franken et al. in [7] (see §1.5, pp. 44). Here, the point process is in discrete time and is given by the sequence $t_n = nN,\ n \in \mathbb{Z}$ and the cycles are the subsequences $\{X_{nN+m+1}^k - X_{nN+m}^l,\ 1 \le k, l \le K,\ 0 \le m < N\}, n \in \mathbb{Z}$. Owing to this representation, it immediately follows from classical Palm inversion arguments, that on the time stationary probability space $(\Omega, F, \hat{\mathcal{P}})$ associated with the synchronous probability space $(\Omega, F, \mathcal{P})$, the stochastic sequence $\{X_n^k - X_{n-1}^l,\ 1 \le i, j \le K\}_{n=-\infty}^{\infty}$ can indeed be made stationary and ergodic with respect to the initial shift θ, so that Equation (6.29) is satisfied.

It is easy to prove that in fact $\mathcal{P} = \hat{\mathcal{P}}$. Let A be in F. The Palm inversion formula ([1] (4.1.2a), p. 13) reads indeed

$$\begin{aligned} N\hat{\mathcal{P}}[A] = &\ \textstyle\sum_0^\infty \mathcal{P}[N > n, \theta^n \in A] \\ = &\ \textstyle\sum_0^{N-1} \mathcal{P}[\theta^n \in A] \\ = &\ N\mathcal{P}[A] \end{aligned} \qquad (6.34)$$

This shows in particular that if $Y^k,\ 1 \le k \le K$ is chosen as any finite solution of Equation (6.27) then, the sequence $\{X_{n+1}^k - X_n^j,\ 1 \le j, k \le K\}_{n=0}^{\infty}$ is in its stationary regime. The continuation to a bi-infinite stationary and ergodic sequence is obtained as above.

The integrability of $M_\infty^{i,j},\ 1 \le i, j \le K$ (see Lemma 7) entails the integrability of the stationary sequence $X_{(n+1)N}^j - X_{nN}^j,\ 1 \le i, j \le K,\ n \in \mathbb{Z}$ (see 6.30). The integrability of the stationary sequence $X^j_{n+1} - X^j_n$, $1 \leq i, j \leq K$, follows directly from this and from Equation (6.33).

In order to prove the property for all components $G_h, 1 \leq h \leq g_0$, one can use the same arguments for the joint processes when choosing for N the Smallest Common Multiple of the integers $N_1, \cdots, N_{g_0}$ associated with these components. □.

We conclude this section by some words on the *proper intensity* of the minimal elements. One gets by induction the relation

$$X^k_n = \max_{\{1 \leq j \leq K\}} \{Y^j + C^{j,k}_{0,n}\} \quad , 1 \leq k \leq K, \; n = 0, 1, 2, \cdots \tag{6.35}$$

where C was defined in Equation (6.2). Let

$$X_n(h) = \max_{\{t \in G_h\}} X^t_n, \quad = \max_{1 \leq k \leq K} X^k_n, \qquad n = 0, 1, 2, \cdots, \quad 1 \leq h \leq g_0 \tag{6.36}$$

Lemma 8 *Let $X_n(h)$ and X^k_n be the sequences associated with subnetwork G_h, $1 \leq h \leq g_0$. There exists a constant μ_h such that*

$$\lim_{n \to \infty} \frac{X_n(h)}{n} = \lim_{n \to \infty} \frac{X^k_n}{n} = \mu_h \quad a.s. \; 1 \leq k \leq K. \tag{6.37}$$

Proof.

Denote by $\hat{X}^k_n$ and $\hat{X}_n(h)$ the value of X^k_n and $X_n(h)$ respectively when $Y^k = 0$. Observe that

$$0 \leq E[\hat{X}_n(h)] < \infty \tag{6.38}$$

where the finiteness of $E[\hat{X}_n(h)]$ follows from the integrability assumptions (use the fact that $\max(a, b) \leq a + b$)) and from the strong connectedness property. Now let $u_{m,m+n} = \hat{X}_n(h) \circ \theta^m$, $m \in \mathbb{Z}$, $n \geq 0$. Then for all $n \geq 1$, and all $1 \leq p < n$, it is easily established that

$$u_{m,m+n} \leq u_{m,m+p} + u_{m+p,m+n}$$

and $u_{m,m+n}$ is hence a non-negative sub-additive process. Applying Kingman's Theorem on sub-additive ergodic processes ([10]) readily yields

$$\lim_{n\to\infty} \frac{u_{0,n}}{n} = \lim_{n\to\infty} \frac{E[u_{0,n}]}{n} = \mu_h \qquad a.s.$$

for some constant μ_h, which concludes the proof for $\hat{X}_n(h)$. The corresponding properties for $\hat{X}_n^k$, follow immediately from this plus the strong connectedness assumption.

For Y^k non zero, (6.37) follows immediately from the bounds

$$\hat{X}_n(h) \leq X_n(h) \leq \max_{\{1\leq k\leq K\}} Y^k + \hat{X}_n(h), \quad n = 0, 1, 2, \cdots \tag{6.39}$$

and from the strong connectedness assumption.□

In the sequel, the constant μ_h^{-1} will be referred to as the *proper intensity* of the component G_h, $1 \leq h \leq g_0$.

7 Ergodicity and Stability of Non-Minimal Elements of the Reduced Graph

7.1 First Step

We consider first the maximal strongly connected subgraph G_h, $h = g_0 + 1$, which allows one to assume that all components T_f, $f < h$ are in their stationary and ergodic regime by an appropriate choice of the initial condition.

Consider the set of components G_f, $f \in \pi(h)$, namely the set of components that are predecessors of G_h in the reduced graph. Denote by f^+ (resp. f^-) the index of any component with the largest (rep. smallest) proper intensity

$$\mu_{f^+} = \min_{f\in\pi(h)} \mu_f \tag{7.1}$$

and

$$\mu_{f^-} = \max_{f \in \pi(h)} \mu_f \tag{7.2}$$

When limited to the components of G_h, the evolution equation in its reduced form reads

$$\begin{aligned} X_0^t = & \quad Y^t, \qquad t \in T_h \\ X_{n+1}^t = & \quad \max\{\max_{\{s \in T_h\}}\{X_n^s + \lambda_n^{s,t}\}, \\ & \max_{\{s \in T_f,\ f < h\}}\{X_n^s + \lambda_n^{s,t}\}\}, \quad n = 0, 1, 2, \cdots,\ t \in T_h \end{aligned} \tag{7.3}$$

For sake of easy notation, we shall again denote by $1, \cdots, K$ the transitions of T_h so that Equations (7.3) read

$$\begin{aligned} X_0^k = & \quad Y^k, \qquad 1 \leq k \leq K \\ X_{n+1}^k = & \quad \max\{\max_{\{1 \leq j \leq K\}}\{X_n^j + \lambda_n^{j,k}\}, \\ & \max_{\{f \in \pi(h)\}} \max_{\{s \in S_f\}}\{X_n^s + \lambda_n^{s,k}\}\}, \quad n = 0, 1, 2, \cdots,\ 1 \leq k \leq K \end{aligned} \tag{7.4}$$

where

$$S_f = \{s \in T_f,\ f \in \pi(h)\ \exists i \in G_h \text{ and } p \in P^1 \text{ s.t. } \gamma^-(p) = s \text{ and } \gamma^+(p) = i\} \tag{7.5}$$

(P^1 was defined in §5).

Denote by μ_h^{-1} the proper intensity of the strongly connected component G_h. Here, the constant μ_h is defined as follows: assume that the constraints due to the components $G_f,\ f \in \pi(h)$ are always satisfied, namely $X_n^s = -\infty, n = 0, 1, \cdots,\ s \in T_f,\ f \in \pi(h)$; then the evolution equation 7.4 of G_h reduces to

$$\begin{aligned} X_0^k = & \quad Y^k, \qquad 1 \leq k \leq K \\ X_{n+1}^k = & \quad \max_{\{1 \leq j \leq K\}}\{X_n^j + \lambda_n^{j,k}\}, \quad n = 0, 1, 2, \cdots,\ 1 \leq k \leq K \end{aligned} \tag{7.6}$$

We are then within the framework considered earlier in §6 and the constant μ_h is defined as in Lemma 8. Observe that this constant is indeed proper to G_h in the sense that it only depends on the matrices $\lambda_n^{j,k},\ j,k \in G_h$.

We first transform the evolution equation 7.4 so as to get rid of the variables $\lambda_n^{j,k}$ and $\lambda_n^{s,k}$ that are equal to $-\infty$.

Let $V_n^k = X_{nM}^k,\ n \geq 0$, for some integer $M \geq 1$. One checks immediately that the variables V_n^k satisfy the recursion

$$
\begin{aligned}
V_0^k = & \quad Y^k, \\
V_{n+1}^k = & \quad \max\{\max_{\{1\leq j\leq K\}}\{V_n^j + \Lambda_n^{j,k}\}, \\
& \max_{\{0\leq m<M\}} \max_{\{1\leq f\in\pi(h)\}} \max_{\{s\in S_f\}}\{X_{nM+m}^s + B_{m,n}^{s,k}\}\}, \\
& \quad n = 0,1,2,\cdots,\ 1 \leq k \leq K
\end{aligned}
\tag{7.7}
$$

where the variables $\Lambda_n^{j,k}$ and $B_{m,n}^{s,k}$ are defined by

$$
\Lambda_n^{j,k} = C_{nM,(n+1)M}^{j,k}, \quad 1 \leq j \leq K, \quad n = 0,1,2,\cdots \tag{7.8}
$$

$$
B_{n,m}^{s,k} = \max_{\{1\leq j\leq K\}}\{\lambda_{nM+m}^{s,j} + C_{nM+m+1,(n+1)M}^{j,k}\}, \tag{7.9}
$$

$$
s \in S_f,\ f \in \pi(h),\ 1 \leq k \leq K,\ n = 0,1,\cdots,\ 0 \leq m < M
$$

where the function C is defined in Equation (6.2).

We use now that fact that G_f is a minimal element of the Reduced Graph to reexpress the variables $X_{nM+m}^s,\ s \in S_f$ that show up in the evolution equation (7.7), in terms of the state variables of G_f, namely $X_n^s,\ s \in T_f$.

For all integer N such that $N < nM$, we have

$$
X_{nM+m}^s = \max_{\{t\in T_f\}}\{X_{nM-N}^t + C_{nM-N,nM+m}^{t,s}\} \tag{7.10}
$$

Let

$$D_n^{t,k} = \max_{\{0\leq m<M\}} \max_{\{s\in T_f\}} \{C^{t,s}_{(nM-N)\bigvee 1,nM+m} + B^{s,k}_{n,m}\}, \quad n = 0,1,2,\cdots \tag{7.11}$$

We get the following final form for the evolution equation(7.7):

Lemma 9 *The variables*

$$V_n^k = X_{nM},\ 1 \leq k \leq K \tag{7.12}$$

and

$$V_n^s = X^s_{(nM-N)\bigvee 1},\ s \in T_f,\ f < h \tag{7.13}$$

satisfy the following evolution equations:

$$\begin{aligned} V_0^k = &\quad Y^k, \\ V_{n+1}^k = &\quad \max\{\max_{\{1\leq j\leq K\}}\{V_n^j + \Lambda_n^{j,k}\}, \max_{\{f\in\pi(h)\}} \max_{\{t\in T_f\}}\{V_n^t + D_n^{t,k}\}\}, \\ &\quad n = 0,1,2,\cdots,\ 1 \leq k \leq K \end{aligned} \tag{7.14}$$

and, for all $f \in \pi(h)$,

$$\begin{aligned} V_0^t = &\quad Y^t,\ t \in T_f,\ n \leq N^* \\ V_{n+1}^t = &\quad \{\max_{\{s\in T_f\}}\{V_n^s + \Lambda_n^{s,t}\},\ n \geq N^*,\ t \in T_f \end{aligned} \tag{7.15}$$

where N^* *is the integer part of* $\frac{N+1}{M}$ *and where the variables* $\Lambda_n^{j,k}$, $\Lambda_n^{s,t}$ *and* $D_n^{t,k}$ *were defined in Equation (7.8), (6.5) and (7.11) respectively.*

Furthermore, for an appropriate choice of the integers M *and* N, *the variables* $\Lambda_n^{j,k}$, $n \geq 0$, $\Lambda_n^{s,t,\ n\geq 0}$ *and* $D_n^{t,k}$, $n \geq N^*$ *are all non-negative and stationary.*

Proof

Equation (7.14) follows immediately from (7.7) and (7.10) while Equation (7.15) is a mere rewriting of (6.4).

Owing to the strong connectedness assumption, one can find an integer M such that the following properties are satisfied:

1. For all $1 \leq j, k \leq K$ and $n = 0, 1, \cdots$, $\Lambda_n^{j,k} \geq 0$;
2. For all $1 \leq k \leq K$ and $s \in S_f$, $f \in \pi(h)$, there is an integer $1 \leq m \leq M$ such that $B_{n,m}^{s,k} \geq 0$.

The proof of the first property is the same as for minimal elements. In order to prove the second fact, observe first that for all $s \in S_f$, there is at least a j, $1 \leq j \leq K$ such that $\lambda_n^{s,j} \geq 0$, $n = 1, 2 \cdots$. Furthermore, if M is large enough, for all $1 \leq l \leq K$, there exists a sequence $j = i_1, i_2, \cdots, i_{M+1} = k$ in $\{1, \cdots, K\}$ such that for all $l = 1, \cdots, M$, $(i_l, i_{l+1}) \in E^*$, so that $\lambda_{nM+l}^{i_l, i_{l+1}} \neq -\infty$. This completes the proof of the second property.

It is clear that owing to the strong connectedness of G_f, one can then chose N large enough for ensuring that the RV's $D_n^{t,k}$, $1 \leq k \leq K$, $f \in \pi(h)$ and $t \in T_f$ are all non-negative (i.e. non $-\infty$), at least for $n \geq \frac{N+1}{M}$, which completes the proof. Observe that the integers N that were defined above and in §6 are not necessarily the same.□

We are now in a position to define the state variables of interest, namely

$$W_n^{t,k} = V_{n+1}^k - V_n^t, \; 1 \leq k \leq K, \; t \in T_e, \; e \in \pi(h), \; n = 0, 1, \cdots \tag{7.16}$$

The basic evolution equations for these variables are given by the following formulae.

Lemma 10 *For all* $1 \leq k \leq K$ *and* $t \in T_f$, $f \in \pi(h)$,

$$\begin{aligned} W_n^{t,k} &= V_{n+1}^k - Y^t, \qquad 0 \leq n \leq N^* \\ W_{n+1}^{t,k} &= \max\{\max_{\{1 \leq j \leq K\}} \min_{\{s \in T_f,\; f \in \pi(h)\}} \{W_n^{s,j} + \Xi_n^{s,j,t,k}\}, \\ &\qquad \max_{\{e \in \pi(h)\}} \max_{\{s \in T_e\}} \{V_{n+1}^s - V_{n+1}^t + D_{n+1}^{s,k}\}\} \qquad n \leq N^* \end{aligned} \tag{7.17}$$

where

$$\Xi_n^{s,j,t,k} = \Lambda_{n+1}^{j,k} - \Lambda_n^{s,t} \tag{7.18}$$

Proof

We get from (7.14) and (7.15) that for all $1 \leq k \leq K$ and $t \in T_f,\ f \in \pi(h)$, we have

$$\begin{aligned} W_n^{t,k} = &\quad V_{n+1}^k - Y^k, \qquad\qquad 0 \leq n \leq N^* \\ W_{n+1}^{t,k} = &\quad \max\{\max_{\{1\leq j\leq K\}}\{V_{n+1}^j + \Lambda_{n+1}^{j,k} - V_{n+1}^t\}, \\ &\max_{\{e\in\pi(h)\}}\max_{\{s\in T_e\}}\{V_{n+1}^s + D_{n+1}^{s,k} - V_{n+1}^t\}\},\quad n \geq N^* \end{aligned} \tag{7.19}$$

and we get Equation (7.17) when that making use of Equation (7.15) in the last relation. □.

Assume that the variables $V_n^s - V_n^t,\ s,t \in T_f,\ f \in \pi(h)$ that show up in Equation (7.17) are all stationary, so that the variables

$$\Psi_n^{t,k} = \max_{\{e\in\pi(h)\}}\max_{\{s\in T_e\}}\{V_{n+1}^s + D_{n+1}^{s,k} - V_{n+1}^t\}, \tag{7.20}$$

are also stationary, at least for $n \geq N^*$. Let $\Theta = \theta^M$ and $\Psi^{t,k}$ be the variable that arises in the stationary process $\Psi_n^{t,k}$, namely the variable such that

$$\Psi_n^{t,k} = \Psi^{t,k} \circ \Theta^n, n \geq N^* \tag{7.21}$$

Let

$$\Xi^{s,j,t,k} = \Xi_0^{s,j,t,k} \tag{7.22}$$

Then, we shall say that Equation (7.17) has a stationary solution if we can find non-negative and finite random variables $W^{t,k},\ 1 \leq k \leq K,\ t \in T_f,\ f \in \pi(h)$ such that

$$W^{t,k} \circ \Theta = \max\{\max_{\{1\leq j\leq K\}}\min_{\{s\in T_e,\ e\in\pi(h)\}}\{W^{s,j} + \Xi^{s,j,t,k}\}, \Psi^{t,k}\} \tag{7.23}$$

In order to find such a solution, consider the sequence $M_n^{t,k}$, $n \geq 0$, $1 \leq k \leq K$, $t \in T_f$, $f \in \pi(h)$ defined by $M_0^{t,k} = 0$, and

$$M_{n+1}^{t,k} \circ \Theta = \max\{\max_{1 \leq j \leq K} \min_{\{s \in T_e,\ e \in \pi(h)\}} \{M_n^{s,j} + \Xi^{s,j,t,k}\}, \Psi^{t,k}\}, \quad n = 0, 1, \cdots \tag{7.24}$$

We prove as in §6 that for all $1 \leq k \leq K$, $t \in T_f$, $f \in \pi(h)$ the sequence $M_n^{t,k}$ is increasing in n and that the limits

$$M_\infty^{t,k} = \lim_{n \to \infty} M_n^{t,k} \tag{7.25}$$

satisfy the following properties:

Lemma 11 *Under the foregoing assumptions, for all* $1 \leq k \leq K$, $t \in T_f$, $f \in \pi(h)$

$$M_\infty^{t,k} \geq 0 \tag{7.26}$$

and

$$M_\infty^{t,k} \circ \Theta = \max\{\max_{1 \leq j \leq K} \min_{\{s \in T_e,\ e \in \pi(h)\}} \{M_\infty^{s,j} + \Xi^{s,j,t,k}\}, \Psi^{t,k}\} \tag{7.27}$$

Furthermore, M_∞ is the minimal solution of Equation (7.17) in the sense that any other solution W satisfies the relation

$$W^{t,k} \geq M_\infty^{t,k} \tag{7.28}$$

However, in the present case, nothing ensures that the limiting variables $M_\infty^{t,k}$, $1 \leq k \leq K$, $t \in T_f$, $f \in \pi(h)$ are finite. It is the object of the following construction (up to Theorem 3 below) to give the conditions under which this property holds. It will then be shown (Lemma 13) that this finiteness property is in fact equivalent to the stability of the places connecting the transitions of T_f, $f \leq h$.

For this, we come back to Equation (7.7). For all $1 \leq k \leq K$ and $f \in \pi(h)$ let

$$A_n^k(f) = \max_{\{0 \leq m < M\}} \max_{\{s \in S_f\}} \{X_{nM+m}^s + B_{n,m}^{s,k}\} \quad n = 0, 1, \cdots \tag{7.29}$$

From Lemma 8 we get immediately the a.s. limit

$$\lim_{n\to\infty} \frac{A_n^k(f)}{n} = \mu_f \quad a.s. \quad 1 \le f < h \tag{7.30}$$

In view of the choice of M, the variables

$$A_n^k = \max_{f\in\pi(h)} A_n^k(f) \tag{7.31}$$

are in IR^+ (i.e. $\neq -\infty$) for all $1 \le k \le K$. Furthermore, it is easy to check that $A_n^k = A_n^k(f^-)$ for n large enough, so that the following limit holds

$$\lim_{n\to\infty} \frac{A_n^k}{n} = \mu_{f^-} \quad a.s. \tag{7.32}$$

where f^- was defined in Equation (7.2). With this notation, Equation (7.7) reads

$$\begin{aligned} V_0^k = & \quad Y^k, & 1 \le k \le K \\ V_{n+1}^k = & \max_{\{1\le j\le K\}}\{\{V_n^j + \Lambda_n^{j,k}\}, A_n^k\}, & n = 0,1,2,\cdots,\ 1 \le k \le K \end{aligned} \tag{7.33}$$

For all $1 \le j, k \le K,\ n = 1,2,\cdots$ define the following variables

$$\begin{aligned} \alpha_n^{j,k}(f) = & \ A_n^k(f) - A_{n-1}^j(f) \\ \alpha_n^{j,k} = & \ A_n^k - A_{n-1}^j \\ \beta_n^{j,k} = & \ \Lambda_n^{j,k} - \alpha_n^{j,k} \end{aligned} \tag{7.34}$$

Consider the following set of state variables

$$\hat{W}_n^k = V_{n+1}^k - A_n^k = \min_{\{t\in T_f,\ f\in\pi(h)\}}\{W_n^{t,k} - D_n^{t,k}\},\ 1 \le k \le K,\ n = 0,1,2,\cdots \tag{7.35}$$

From (7.33), it is easily checked that the non-negative RV's $\hat{W}_n^k$ satisfy the recursion

$$\begin{aligned} \hat{W}_0^k = & \max_{\{1\le j\le K\}}\{Y^j + \Lambda_0^{j,k} - A_0^k\}^+, & 1 \le k \le K \\ \hat{W}_{n+1}^k = & \max_{\{1\le j\le K\}}\{\hat{W}_n^j + \Lambda_n^{j,k} - (A_n^k - A_{n-1}^j)\}^+, & 1 \le k \le K,\ n = 1,2,\cdots \end{aligned} \tag{7.36}$$

This type of equation is a slight generalization of the class of evolution equations analyzed in [2]. We state first some basic results on this equation in Lemma 12 and Theorem 2 below.

For $1 \leq k \leq K,\ n = 1, 2, \cdots$, let

$$H_n^k = \max_{\{1 \leq j_0, j_1, \cdots, j_n = k \leq K\}} \{(\sum_{m=0}^{n-1} \Lambda^{j_m, j_{m+1}} \circ \Theta^{-n+m}) + A_n^k \circ \Theta^{-n} - A_0^{j_0} \circ \Theta^{-n}\} \tag{7.37}$$

and

$$H_n = \max_{1 \leq k \leq K} H_n^k, \qquad n = 1, 2, \cdots \tag{7.38}$$

Lemma 12 *Under the foregoing assumptions*

$$\lim_{n \to \infty} \frac{H_n}{n} = \lim_{n \to \infty} \frac{E[H_n]}{n} \lim_{n \to \infty} \frac{H_n^k}{n} = \lim_{n \to \infty} \frac{E[H_n^k]}{n} = \mu_h - \mu_{f-} \quad , \quad a.s., \tag{7.39}$$

Proof.

Using the subadditive property as in Lemma 8 we get

$$\lim_{n \to \infty} \frac{1}{n} \max_{\{1 \leq j_0, j_1, \cdots, j_n = k \leq K\}} \{(\sum_{m=0}^{n-1} \Lambda^{j_m, j_{m+1}} \circ \Theta^{-n+m}) = \mu_h \tag{7.40}$$

Using Equation (7.32) plus the integrability of A_0^j, which entails that $\frac{A_0^j \circ \Theta^{-n}}{n} \to 0,\ 1 \leq j \leq K$ a.s., we get

$$\lim_{n \to \infty} \frac{1}{n}(A_n^k \circ \Theta^{-n} - A_0^j \circ \Theta^{-n}) = \mu_{f-} \quad , \ 1 \leq k \leq K \tag{7.41}$$

Equation (7.39) follows directly from the last two limits after elementary manipulations.□

Consider the following set of assumptions.

H-4

The differences $\delta_n^{s,t} = X_n^s - X_{n-1}^t$ are stationary for all $s \in T_f,\ t \in T_e,\ e, f \in \pi(h)$.

Theorem 2 *Under Assumption H-4, if $\mu_h < \mu_{f^-}$, Equation (7.36) has a unique a.s. finite stationary solution $\hat{M}_\infty^k \circ \theta^n,\ 1 \le k \le K,\ n \in \mathbb{Z}$, where $\hat{M}_\infty^k$ is given by*

$$\hat{M}_\infty^k = \max(0, \max_{1 \le n} H_n^k) \tag{7.42}$$

and whatever the initial condition of Equation (7.36), the sequence $\hat{W}_n^k$ couples within finite time with the stationary process $\hat{M}_\infty^k \circ \theta^n$.

If $\mu_h > \mu_{f^-}$, Equation (7.36) has no finite stationary solution and whatever the initial condition, the sequence $\frac{\hat{W}_n^k}{n}$ tends to $\mu_h - \mu_{f^-}$ a.s., so that $\hat{W}_n^k$ tends to infinity when n tends to infinity.

proof

Under the foregoing assumptions the sequence $A_{n+1}^k - A_n^j,\ n = 0, 1, \cdots$ is stationary for all $1 \le j, k \le K$. Indeed we have

$$\begin{aligned} \alpha_n^{i,j} = \ & \max\nolimits_{\{0 \le m < M,\ s \in T_f,\ f \in \pi(h)\}} \min\nolimits_{\{0 \le p < M,\ t \in T_f,\ f \in \pi(h)\}} \\ & \{X_{nM+m}^s - X_{(n-1)M+p}^t + B_{m,n}^{s,j} - B_{p,n-1}^{t,i}\} \end{aligned} \tag{7.43}$$

Owing to Assumption H-4, for all $0 \le m, p < M,\ s, t \in T_f,\ f \in \pi(h)$, the difference $X_{nM+m}^s - X_{(n-1)M+p}^t$ can be represented as

$$\{X_{nM+m}^s - X_{(n-1)M+p}^t\} = \{X_{M+m}^s - X_p^t\} \circ \theta^{(n-1)M}, \quad n \in \mathbb{Z}$$

One immediately gets from this and from the definition of $B_{m,n}^{s,j}$ that

$$\alpha_n^{i,j} = \alpha^{i,j} \circ \Theta^n,\ n \in \mathbb{Z}$$

where $\alpha^{i,j} = \alpha_0^{i,j}$.

Given this stationarity property, the rest of the proof is the same as the proofs of Theorem 3.8 and Lemma 4.2 in [2]. □

We show first that under the foregoing assumptions, the stability of the variables $\hat{W}$ is in fact equivalent to the stability of the variables W defined in Equation (7.16). More precisely, the following result holds.

Theorem 3 *Under the foregoing assumptions, if $\mu_h < \mu_{f}-$, then the h first components of the network can be made stationary in the sense that for an appropriate choice of the initial conditions, the differences $\delta_n^{s,t} = X_n^t - X_{n-1}^s$, $t \in T_h$, $s \in T_f$, $f \in \pi(h)$ satisfy the relation*

$$\delta_n^{s,t} = \delta^{s,t} \circ \theta^n, \quad n \in \mathbb{Z} \tag{7.44}$$

for some finite random variable $\delta^{s,t}$. If $\mu_h > \mu_{f}-$, then the variables $\delta_n^{s,t}$, $s \in T_f$, $f \in \pi(h)$, $t \in T_h$ converge to ∞ a.s..

Proof

If $\mu_h < \mu_{f}-$, we know that for all $1 \leq k \leq K$, the sequence

$$\begin{aligned} \hat{W}_n^k = & \quad \min_{\{t \in T_f,\ f \in \pi(h)\}} \{W_n^{t,k} - D_n^{t,k}\} \\ = & \ \min_{\{t \in T_f,\ f \in \pi(h)\}} \{V_{n+1}^k - V_n^t - D_n^{t,k}\} \end{aligned} \tag{7.45}$$

couples within finite time with a finite stationary sequence. In addition, owing to Assumption H-4, the differences $V_n^s - V_n^t$, $s \in T_f$, $t \in T_e$, $e, f \in \pi(h)$ are stationary and finite thanks to an appropriate choice of the initial condition. This together with the fact that $D_n^{t,k}$ couples with a stationary and finite sequence for $n \geq N^*$ immediately entail that the differences

$$V_n^s - \min_{\{t \in T_f,\ f \in \pi(h)\}} \{V_n^t - D_n^{t,k}\} \tag{7.46}$$

also couple within finite time with their stationary version. From these two remarks, we deduce immediately that the differences

$$V_{n+1}^k - V_n^s = \hat{W}_n^k - V_n^s + \min_{\{t \in T_f,\ f \in \pi(h)\}} \{V_n^t - D_n^{t,k}\} \tag{7.47}$$

do also couple with a stationary and finite process within finite time. Hence, the differences $\Delta_n^{s,k} = V_{n+1}^k - V_n^s$ can be made $\Theta = \theta^M$ stationary in the usual way.

On the other hand if $\mu_h > \mu_{f^-}$, we know from Theorem 2 that $\hat{W}_n^k$ tends to ∞ a.s.. This together with the bound

$$\hat{W}_n^k \leq W_n^{t,k},\ t \in T_f,\ f \in \pi(h),\ 1 \leq k \leq K \tag{7.48}$$

show that in this case, $W_n^{t,k}$ tends to ∞ a.s. for all $t \in T_f,\ f \in \pi(h)$ and $1 \leq k \leq K$.

The rest of the proof, namely the relation between the Θ stationarity of $V_{n+1}^k - V_n^t$ and the θ stationarity of $X_{n+1}^k - X_n^t$ is treated as in Theorem 1. $\square$.

One understands from Theorem 2 that the first basic property to check is the *stationarity* of the sequences $X_n^s - X_n^t$, $s \in T_f,\ t \in G_e,\ e, f \in \pi(h)$.

We focus first on the case where $h = g_0 + 1$ has a unique predecessor f in the Reduced Graph. Then Assumption H-4 is a direct consequence of Theorem 1 and if $\mu_h < \mu_f$, one gets immediately from Theorem 2 that the RV's $W_n^{s,k}$ couple within finite time with the stationary process $M_\infty^{s,k} \circ \Theta^n$. On the other hand, if $\mu_h > \mu_f$, $W_n^{s,k}$ tends a.s. to ∞ for all $1 \leq k \leq K$.

We consider now the case where there are several strongly connected components preceding $h = g_0 + 1$.

If these components have different proper intensities, it is easy to check that $A_n^k = A_n^k(f^-)$ for n large enough. This plus elementary coupling arguments allow one to reduce the analysis of $\hat{W}$ to the case with only one predecessor. However, it is clear that differences of the type $V_n^k - V_n^s$, which can be rewritten as $\hat{W}_n^k + A_n^k - V_n^s$ for n large enough, tend a.s. to ∞ for all $1 \leq k \leq K$, $s \in G_{f+}$.

If all the predecessors have the same proper intensity, it turns out that nothing general (namely independent on higher statistics) can be said on the stationarity of the variables $X_n^s - X_n^t$ for $s \in T_f$, $t \in G_e$, $e \neq f \in \pi(h)$. An example of this fact is given in the appendix.

A few remarks are in order.

Remark 1

It is not always true that the stationary differences $X_{n+1}^k - X_n^t$ are integrable. Simple counter examples can be found in Queueing Theory. For instance, the stationary waiting times in the $GI/GI/1$ queue fall in that category of differences and are only integrable under some conditions on the second moments of the constituting sequences.

when n goes to ∞.

Remark 2

Consider the following problem: assume that the constant μ_h is fixed; what is the maximal value of μ_f for which the first h components of the network are stable? The stability condition $\mu_h < \mu_{f-}$ shows that the *waiting times* in places connecting transitions of T_h to transitions of T_f, $f \in \pi(h)$ get unstable when μ_f is equal to μ_h. This result is a natural extension of Lavenberg's Theorem [11], in that it states that the stability condition of the network consists in requiring that the proper intensity of the upstream component(s) be less than the proper intensity of G_h when saturating all the constraints due to G_f, $f \in \pi(h)$.

We conclude this section by a lemma showing the equivalence between the instability of waiting times and the instability of places as defined in §5.

Lemma 13 *Under the assumptions of Theorem 3, if $\mu_h < \mu_{f^-}$, then the h first components of the network are stable in the sense that the places of the SEG connecting transitions of T_f, $f \leq h$ have all a finite marking in steady state. Conversely, if $\mu_h > \mu_{f^-}$, then theses places are all unstable whatever the initial conditions.*

Proof

Let p be a place of P^1, $s = \gamma^-(p) \in S_f$ and $j = \gamma^+(p)$, $1 \leq j \leq K$.

We analyze first the case $\mu_h < \mu_{f^-}$. Consider the network to be in steady state, due to an appropriate choice of the initial condition. Owing to the FIFO assumptions, $N^j(p)$, the number of tokens in place p at time X_1^j is exactly

$$N^j(p) = \sum_{m=1+M(p)}^{\infty} I_{\{X_n^s \leq X_1^j\}} \tag{7.49}$$

which is obviously finite since $X_1^j < \infty$ and since X_n^s tends to ∞ a.s. (use the fact that $\frac{X_n^s}{n}$ tends to μ_f) when n goes to ∞.

Conversely, consider any set of initial conditions. For p as above, let $Q_n(p)$ denote the number of tokens in p at time X_n^s. The m-th token to enter place p will still be there at time X_n^s iff $X_m^j \geq X_n^s$. We have hence

$$Q_n(p) = \sum_{1 \leq m \leq n+M(p)} I_{\{X_{n-m}^j - X_n^s \geq 0\}} \tag{7.50}$$

If $\mu_h > \mu_{f^-}$, we know from Theorem 2 that the differences $\frac{X_m^j - X_n^s}{n-m}$ tend to $\nu = \mu_h - \mu_{f^-}$ a.s.. This entails that for m fixed, $\forall \epsilon$, $\exists N$ such that $\forall n \geq N$, $X_m^j - X_n^s \geq (n-m)(\nu + \epsilon)$.

It follows immediately from this that $I_{\{X^j_m - X^s_n \geq 0\}}$ is equal to one a.s. for $n \geq N$. One gets immediately from this and from Equation (7.50) that $M_n(p)$ tends to ∞ a.s.

7.2 General Case

Consider now any non minimal element G_h of the Reduced Graph. The stability of G_h can be analyzed as the stability of G_{g_0+1} in §7.1. We will hence limit ourselves to an outline of the main results and of the basic ideas of the proofs.

We denote by $\pi(h)$ (resp. $\pi^*(h)$) the set of direct (resp. direct or indirect) predecessors of h in $\mathcal{G}$ anf by $\pi^0(h)$ the set

$$\pi^0(h) = \{e \in \pi^*(h) \cap \{1, \cdots, g_0\}\} \tag{7.51}$$

In view of the results of §7.1, it will be assumed that there is a single minimal element, namely $g_0 = 1$, so that $\pi^0(h) = \{1\}$, $\forall h = 2, \cdots, g$.

Theorem 4 *Under the foregoing assumptions, if the proper intensities μ_h, $h = 2, 3, \cdots, g$ satisfy the relation*

$$\mu_h < \mu_1, \forall h = 2, 3, \cdots, g \tag{7.52}$$

then, for an appropriate choice of the initial conditions of the minimal element, the variables $\delta^{s,t}_n = X^t_n - X^s_{n-1}$, $s, t \in T$ couple with their stationary version in the sense that from a finite rank

$$\delta^{s,t}_n = \delta^{s,t} \circ \theta^n, \tag{7.53}$$

for some finite random variable $\delta^{s,t}$. If $\mu_h > \mu_1$, for some $h = 2, 3, \cdots, g$, then the variables $\delta^{s,t}_n$, $s \in T_f$, $f \in \pi(h)$, $t \in T_h$ converge to ∞ a.s..

Proof

This proof is based on an induction on the index $h = 2, \cdots, g$. Consider the following

Induction Assumption

Each of the variables $\delta_n^{s,t}$, $t \in T_f$, $f \in \pi(h)$, $s \in T_1$ couples with its steady state version within finite time.

It is satisfied for $h = 3$ if all minimal elements are assumed to be in their steady state (see Theorem 3).

Assume the induction assumption to hold up to rank $h-1$. Observe that this assumption entails that

$$\lim_{n \to \infty} \frac{X_n^t}{n} = \mu_1, \quad \forall t \in \pi(h) \tag{7.54}$$

Adopting the same notations as in §7.1, Equation (7.4) remains valid. Define

$$V_n^k = X_{nM}, \ k \in \{1, \cdots, K\} = G_h \tag{7.55}$$

and

$$V_n^t = X^t_{(nM-N)\bigvee 1}, \ t \in T_f, \ f \in \pi(h) \tag{7.56}$$

We get as in Lemma 9 that these variables satisfy the following evolution equations:

$$\begin{aligned} V_0^k = & \quad Y^k, \\ V_{n+1}^k = & \ \max\{\max_{\{1 \le j \le K\}}\{V_n^j + \Lambda_n^{j,k}\}, \max_{\{f \in \pi(h)\}} \max_{\{t \in T_f\}}\{V_n^t + D_n^{t,k}\}\}, \\ & n = 0, 1, 2, \cdots, \ 1 \le k \le K \end{aligned} \tag{7.57}$$

where the variables Λ and D are defined as in §7.1. Similarly, the variables

$$V_n^s = X_{nM}^s, \quad s \in T_1 \tag{7.58}$$

satisfy an evolution equation of the type (6.4), namely

$$\begin{aligned} V_0^s = & \quad Y^s, \qquad s \in T_1 \\ V_{n+1}^s = & \ \max_{\{u \in T_1\}}\{V_n^u + \Lambda_n^{u,s}\}, \ n = 0, 1, 2, \cdots, \ s \in T_1 \end{aligned} \tag{7.59}$$

From this, we get that the state variables

$$W_n^{s,k} = V_{n+1}^k - V_n^s, \ 1 \le k \le K, \ t \in T_1, \ n = 0, 1, \cdots \qquad (7.60)$$

satisfy the evolution equations

$$W_0^{s,k} = \qquad V_1^k - Y^s, \qquad (7.61)$$

$$W_{n+1}^{s,k} = \quad \max\{\max_{\{1\le j\le K\}} \min_{\{u\in T_1\}}\{W_n^{u,j} + \Xi_n^{u,j,t,k}\},$$

$$\max_{\{f\in\pi(h)\}} \max_{\{t\in T_f\}}\{V_{n+1}^t - V_{n+1}^s + D_{n+1}^{t,k}\}\} \quad n \le 0 \qquad (7.62)$$

where

$$\Xi_n^{u,j,t,k} = \Lambda_{n+1}^{j,k} - \Lambda_n^{u,s} \qquad (7.63)$$

Based on the induction assumption, we get the following results exactly as in Theorems 2 and 3 in §7.1:

If $\mu_h < \mu_1$, then the differences $\delta_n^{s,k}, \ k \in T_h, \ s \in T_1$ couple within finite time with their stationary version too, namely from a certain rank,

$$\delta_n^{s,k} = \delta^{s,k} \circ \theta^n, \ k \in T_h, \ s \in T_1 \qquad (7.64)$$

where $\delta^{s,k}$ is some finite random variable. From that, it is easily checked that more generally, the RV's $\delta_n^{t,k}, \ t \in T_f, \ f \in \pi^*(h), \ k \in T_h$ do also couple with their steady state version within finite time.

On the other hand, if $\mu_h > \mu_1$, then the variables $\delta_n^{s,k}, \ s \in T_1, \ k \in T_h$ converge to ∞ a.s..□

One gets as in Lemma 13 that the places of the network will be stable if Equation (7.52) is satisfied and that if h_0 is the first integer such that $\mu_h > \mu_1$, then the places connecting transitions of $G_f, \ f \in \pi(h_0)$ to transitions of G_{h_0} will be unstable.

Fore more general topologies, similar results can be derived by propagating the induction assumption

Induction Assumption

Each of the variables $\delta_n^{s,t}$, $t \in T_f$, $f \in \pi(h)$, $s \in T_e$, $e \in \pi^0(h)$ *couples with its steady state version within finite time.*

But this requires more information on the statistics as it was mentioned §7.1 and exemplified in the appendix.

8 Appendix

Consider a Timed Event Graph with three recycled transitions s_1, s_2, and t and five places p_1, p_2, q_1, q_2 and r. p_i (resp. r) is the place associated with the recycling of s_i, $i = 1, 2$ (resp. t) and q_i is the place connecting s_i to t. More precisely

$$\begin{aligned}
\gamma^-(s_i) &= p_i, \quad \gamma^+(s_i) = \{p_i, q_i\}, \quad i = 1, 2 \\
\gamma^-(t) &= \{r, q_1, q_2\} \qquad \gamma^+(t) = r \\
\gamma^-(p_i) &= s_i, \qquad \gamma^+(p_i) = s_i, \qquad i = 1, 2 \\
\gamma^-(q_i) &= s_i, \qquad \gamma^+(q_i) = t \qquad i = 1, 2 \\
\gamma^-(r) &= t \qquad \gamma^+(r) = t
\end{aligned} \tag{8.1}$$

Within the terminology of [3], this system is a *Join* queue with one server (transition t) and with two sources (transitions s_1 and s_2). Within the terminology of §4, we have two minimal components G_i, with $T_i = \{s_i\}$ and $E_i = (s_i, s_i)$, $i = 1, 2$ and one non minimal component G_3, with $T_3 = \{t\}$ and $E_3 = (t, t)$.

Consider the simple case where the service times in r, q_1 and q_2 are zero and where the service times in p_1 and p_2 are independent i.i.d sequences σ_n^1 and σ_n^2, with common mean λ^{-1}. If the two sequences σ_n^1 and σ_n^2 are deterministic, it is easily checked that the differences $X_n^{s_1} - X_n^{s_2}$ are stationary and finite whatever the initial conditions. However, if the two sequences are made of exponentially distributed RV's with common parameter

λ, the differences $X_n^{s_1} - X_n^{s_2}$ form a recurrent null Markov Chain on $I\!R$ that admits no invariant measure with finite mass, so that they cannot be made stationary.

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